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New Kinetic Equation for Pair-annihilating Particles: Generalization of the Boltzmann Equation

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ABSTRACT

A convenient form of kinetic equation is derived for pair annihilation of heavy stable particles relevant to the dark matter problem in cosmology. The kinetic equation thus derived extends the on-shell Boltzmann equation in a most straightforward way, including the off-shell effect. A detailed balance equation for the equilibrium abundance is further analyzed. Perturbative analysis of this equation supports a previous result for the equilibrium abundance using the thermal field theory, and gives the temperature power dependence of equilibrium value at low temperatures. Estimate of the relic abundance is possible using this new equilibrium abundance in the sudden freeze-out approximation.

1 Introduction

Generalization of the Boltzmann equation is important both from the point of fundamental physics and of application to many areas of physics. Since its investigation involves quantum dynamics at finite time, its general formulation in a form of practical use is undoubtedly difficult. One important classs of physical situations however seems amenable to relatively straightforward analysis. It is a small system immersed in a large thermal environment.

Application of this class of physical situations includes pair-annihilation of heavy stable particles in cosmology; the dark matter problem [1]. The conventional practice for estimate of the relic abundance of dark matter particles is first to compute the freeze-out temperature below which the annihilation effectively ceases due to rapid cosmological expansion, and then to use the ideal gas form of number density at that temperature. A thermally averaged Boltzmann equation is a basis for such computations [2]. Clearly, the use of the on-shell Boltzmann equation and the ideal gas distribution function should be questioned, especially at low temperatures such as $T < M/30$ (M being the mass of the heavy particle) with very small $e^{-M/T}$ like 10^{-13}, which happens to be the relevant temperature region for the dark matter annihilation.

As in our previous studies [3],[4],[5], we take in the present work of quantum kinetic equation the method of integrated-out environment in much the same way as some treatment of quantum Brownian motion [6]. This way one can well cope with quantum mechanical behavior at finite times beyond the infinite time behavior taken care of in the S-matrix approach of the Boltzmann equation. In this paper we report a great and practical simplification over our previous published result [3]. We are thus ready for a realistic study of supersymmetric dark matter problem.

The equilibrium number density thus computed agrees with that obtained using the thermal field theory in [7]. The result of the present work is however more general; we present the full quantum kinetic equation, not just the equilibrium result. The essential ingredient in this formulation is the Hartree approximation, which makes it possible to use the result of the solvable model [4], with an extra assumption of slow variation of physical quantities.

2 Model of pair annihilation

Suppose that a pair of heavy particles φ (boson in this case) can annihilate into a χ (boson) pair; $\varphi\varphi \rightarrow \chi\chi$ with a dimensionless interaction strength λ. A discrete symmetry under $\varphi \rightarrow -\varphi$ is imposed to forbid the φ decay, $\varphi \rightarrow \chi\chi$. We assume that the lighter particle χ makes up a thermal environment of temperature $T = 1/\beta$ in our unit of the Boltzmann constant $k_B = 1$. Thermalization becomes possible either due to their own self-interaction characterized by a dimensionless coupling λ_χ, or interaction with other light particles which need not to be specified for our purpose. A relativistic field theory model we consider is thus given by the Lagrangian density

$$\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_\chi + \mathcal{L}_{\rm int}\,, \tag{1}$$

$$\mathcal{L}_\varphi + \mathcal{L}_\chi = \frac{1}{2}\,(\partial_\mu \varphi)^2 - \frac{1}{2}\,M^2\,\varphi^2 + \frac{1}{2}\,(\partial_\mu \chi)^2 - \frac{1}{2}\,m^2\,\chi^2\,, \tag{2}$$

$$\mathcal{L}_{\rm int} = -\,\frac{\lambda}{4}\,\varphi^2\,\chi^2 - \frac{\lambda_\varphi}{4!}\,\varphi^4 - \frac{\lambda_\chi}{4!}\,\chi^4 + \delta\mathcal{L}\,. \tag{3}$$

The last two terms ($\propto \lambda_\varphi$ and λ_χ) in the interaction are introduced both for consistency of renormalization and for thermalization of χ particles, hence the parameters are taken to satisfy $|\lambda_\varphi| \ll \lambda^2 \ll 1\,, \quad |\lambda_\chi| < 1\,,$ but $|\lambda_\chi| \gg |\lambda|$. The renormalization counter term $\delta\mathcal{L}$ is given by

$$\frac{\delta Z_\varphi}{2}(\partial_\mu\varphi)^2 - \frac{\delta M^2}{2}\varphi^2 + \frac{\delta Z_\chi}{2}(\partial_\mu\chi)^2 - \frac{\delta m^2}{2}\chi^2 - \frac{\delta\lambda}{4}\varphi^2\chi^2 - \frac{\delta\lambda_\varphi}{4!}\varphi^4 - \frac{\delta\lambda_\chi}{4!}\chi^4\,. \tag{4}$$

We focuss on a particular dynamical degree of freedom, the heavy φ field, and integrates out the environment part, the χ field, altogether. In the influence functional method [8] this integration is carried out for the squared amplitude, namely for the probability function, by using the path integral technique. This way one has to deal with the conjugate field variable φ' along with φ, since the complex conjugated quantity is multiplied in the probability. The influence functional $\mathcal{F}$ is thus defined by

$$\int \mathcal{D}\chi \int \mathcal{D}\chi'\, \exp\Big[i \int dx\, \big(\, \mathcal{L}_\chi(x) - \mathcal{L}_{\chi'}(x) + \mathcal{L}_{\rm int}(\varphi(x)\,, \chi(x)) - \mathcal{L}_{\rm int}(\varphi'(x)\,, \chi'(x))\,\big) \Big]\,. \tag{5}$$

We convolute with the influence functional the initial and the final density matrix of the χ system. For the initial state it is assumed that the entire system is described by an uncorrelated product of the system and the environment density matrix,

$$\rho_i = \rho_i^{(\varphi)} \times \rho_i^{(\chi)} \,, \quad \rho_i^{(\chi)} = \rho_\beta^{(\chi)} = e^{-\,\beta H_0(\chi)}/\mathrm{tr}\, e^{-\,\beta H_0(\chi)} \,. \tag{6}$$

Here $\rho_\beta^{(\chi)}$ is the density matrix for a thermal environment. The density matrix $\rho_i^{(\varphi)}$ for the φ system is arbitrary. The choice of the initial density matrix ρ_i is not crucial and the final result is expected to be insensitive to this choice provided that it does not commute with the total hamiltonian, $[\rho_i \,, H] \neq 0$. We need this condition to guarantee a departure from complete thermal equilibrium. At a final time t_f the χ integration is performed taking the condition of non-observation for the environment,

$$\int d\chi_f \int d\chi_f'\, \delta(\chi_f - \chi_f')\, (\cdots) \,, \tag{7}$$

with the understanding that the environment is totally unspecified at the time t_f.

The result of χ integration is given by a series of Gaussian integral if we expand in powers of λ_χ of the $\lambda_\chi\, \chi^4$ interaction, and the influence functional to order λ^2 is of the form,

$$\mathcal{F}_4[\varphi \,, \varphi'] =$$

$$\exp[\,-\,\frac{1}{4}\int_{x_0 > y_0} dx\, dy\, \left(\, \xi_2(x)\, \alpha_R(x\,, y)\, \xi_2(y) + i\, \xi_2(x)\, \alpha_I(x\,, y)\, X_2(y) \,\right)] \,, \tag{8}$$

$$X_2(x) \equiv \varphi^2(x) + \varphi'^{\,2}(x) \,, \quad \xi_2(x) \equiv \varphi^2(x) - \varphi'^{\,2}(x) \,, \tag{9}$$

$$\alpha(x\,, y) = \alpha_R(x\,, y) + i\alpha_I(x\,, y)$$

$$= \lambda^2 \,\left(\mathrm{tr}\, \left(T[\chi^2(x)\chi^2(y)\, \rho_\beta^{(\chi)}]\right) - (\,\mathrm{tr}\, \chi^2 \rho_\beta^{(\chi)}\,)^2\,\right) \,, \tag{10}$$

if one replaces the mass term in the original Lagrangian by a $O[\lambda]$ temperature dependent mass, $M^2(T) = M^2 + \frac{\lambda}{24}\, T^2$ valid for $m \ll T$. Note the presence of the time ordering, $x_0 > y_0$, in the above formula. The kernel function $\alpha(x\,, y)$ satisfies the time translation invariance, thus may be written as $\alpha(x - y)$. An explicit form of the kernel function $\alpha(x)$ or its Fourier transform is given later in eqs.(23) and (25). Higher order terms in λ^2 are actually present in the exponent of the influence functional. These contribute either to many-body processes we are not interested in, or to negligible higher order terms to our process.

The convolution with the heavy φ system gives the reduced density matrix at time t_f;

$$\rho^{(R)}(\varphi_f \,, \varphi_f') = \int d\varphi_i \int d\varphi_i' \int \mathcal{D}\varphi \int \mathcal{D}\varphi'\, e^{iS(\varphi) - iS(\varphi')}\, \mathcal{F}_4[\varphi \,, \varphi']\, \rho_i^{(\varphi)}(\varphi_i \,, \varphi_i') \,, \tag{11}$$

from which one can deduce physical quantities for the φ system. Here $S(\varphi)$ is the action for the φ system obtained from the basic Lagrangian.

The correlator is defined using the reduced density matrix; for $x_0 > y_0$

$$\langle\, \varphi(x)\varphi(y)\, \rangle = \int d\varphi(x) \int d\varphi'(x) \int d\varphi(y) \int d\varphi'(y)\, \delta\, (\varphi(x) - \varphi'(x))$$
$$\cdot \int \mathcal{D}\varphi \int \mathcal{D}\varphi'\, e^{iS(\varphi)-iS(\varphi')}\, \mathcal{F}_4[\varphi\,, \varphi']\, \varphi(x)\varphi(y)\, \rho^{(R)}(\varphi(y)\,, \varphi'(y))\,. \tag{12}$$

The path to be taken here is those defined on the time interval, $x_0 > t > y_0$. A nice feature of this correlator formula is that the initial memory effect appears compactly via the reduced density matrix $\rho^{(R)}$. With the Heisenberg evolution $\varphi(x) = e^{iHt}\varphi(\vec{x}\,, 0)e^{-iHt}$, this correlator is equal to

$$\mathrm{tr}\ \rho_i\, \varphi(x)\varphi(y)\,, \tag{13}$$

where ρ_i is the total initial density matrix (6). Whenever this density matrix does not commute with the total hamiltonian, $[\rho_i\,, H] \neq 0$, this correlator does not respect the time-translation invariance,

$$\langle \varphi(\vec{x}\,, t_1)\varphi(\vec{y}\,, t_2)\rangle \neq \langle \varphi(\vec{x}\,, t_1 - t_2)\varphi(\vec{y}\,, 0)\rangle\,, \tag{14}$$

unlike the correlator in complete thermal equilibrium.

The model thus specified is difficult to solve due to the appearance of the quartic term of φ in the influence functional (8). The situation is however simplified when one considers a mean field approximation. In the mean field or the Hartree approximation one replaces an even number of field operators by a pair of two operators times the rest of several averaged two-body correlators. This approximation is good if one can ignore a higher order correlation than that of two-body. The Hartree model is expected to work well in a dilute system characterized by a low occupation number for each mode.

The Hartree approximation we introduce is then a Gaussian truncation to the influence functional; we replace the original one by properly defining new real kernel functions $\beta_i(x\,, y)$ in the quadratic form;

$$\mathcal{F}_2[\varphi\,, \varphi'] = \exp[-\int_{x_0 > y_0} dx\, dy\ (\, \xi(x)\, \beta_R(x\,, y)\, \xi(y) + i\, \xi(x)\, \beta_I(x\,, y)\, X(y)\,)\,]\,, \tag{15}$$
$$X(x) \equiv \varphi(x) + \varphi'(x)\,, \quad \xi(x) \equiv \varphi(x) - \varphi'(x)\,. \tag{16}$$

When the time-translation invariance holds for β_i, one can work out its consequences to all order of λ for a given kernel $\beta_i(x\,, y)$, since it becomes a solvable Gaussian model.

For derivation of the self-consistency relation introduce the full propagator by

$$G(x\,,y) = i\,\langle\varphi(x)\varphi(y)\rangle_{\mathcal{F}=\mathcal{F}_4}\,, \tag{17}$$

using the influence functional (8). We compute correlators,

$$\langle X(x)\xi(y)\rangle\,,\langle X(x)X(y)\rangle\,,\langle\xi(x)\xi(y)\rangle\,, \tag{18}$$

in two theories, using the two different forms of the influence functional, $\mathcal{F}_4$ and the truncated one $\mathcal{F}_2$, and identify two results. One then has ($\beta = \beta_R + i\beta_I$);

$$\beta(x_1\,,x_2) = -\,i\,\alpha(x_1 - x_2)\,G(x_1\,,x_2)\,. \tag{19}$$

It is important to note again that there is no translational invariance with respect to time variables $t_1\,,t_2$, since the system is not in complete thermal equilibrium. Thus, one should distiguish two time variables for the two-point correlator, the relative time $\tau = t_1 - t_2$ and the the central time $t = (t_1 + t_2)/2$. A fundamental strategy in the present work is to note the slow change of physical quantities for the central time and exploit it in the analysis of the above relation (19). One thus solves short time dynamics of the relative time assuming a constant central time. The result of solvable Gaussian model [4], [5] may then be used provided that the kernel function β is given. Since the correlator thus derived is a functional of an yet unknown β, one arrives at the self-consistency equation for this β, which contains the constant central time. This constant central time is then allowed to slowly vary in the spirit of adiabatic approximation. We shall explain this procedure in detail in due course.

3 Correlation function in Hartree approximation

It is convenient to define the Fourier transform with respect to the relative variable,

$$\sigma(k,t) = \int d^4(x-y)\,\langle\varphi(x)\,\varphi(y)\rangle e^{ik\cdot(x-y)}\,. \tag{20}$$

Since spatial homogeneity holds, this quantity σ may depend on the central coordinate only via $t = (x_0 + y_0)/2$.

Suppose that the central time dependence is slow, and we may take this as a constant. The correlation function can then be calculated using the technique of the generating functional for the influence functional, as explained in ref.[3]. In the Gaussian model or in the Hartree model of pair annihilation each Fourier component of correlators is given in terms of a basic function $g(\vec{k}\,,t)$ and the density matrix of the initial state. Suppressing all Fourier momenta and denoting $q(t)$ for Fourier component of field and $p(t)$ for its time derivatives, one has, for instance, for the $q-q$ correlator

$$\langle q(t_1)q(t_2)\rangle = -\frac{i}{2}\,g(t_1-t_2) + \int_0^{t_1} dt \int_0^{t_2} ds\, g(t_1-t)\beta_R(t-s)g(t_2-s)$$
$$+\,g(t_1)g(t_2)\,\overline{p_i^2} + \dot{g}(t_1)\dot{g}(t_2)\,\overline{q_i^2} + \frac{1}{2}\,(\,\dot{g}(t_1)g(t_2) + g(t_1)\dot{g}(t_2)\,)\,\overline{\{\,p_i\,,q_i\,\}}\,. \tag{21}$$

The initial density matrix elements are needed only to sepcify three expectation values, $\overline{p_i^2}\,,\overline{q_i^2}$ and $\overline{\{\,p_i\,,q_i\,\}}$.

The basic function $g(t)$ is related to the spectral weight $r(\omega)$, and ultimately to the Fourier transform of the correlator. Fourier transforms, r and r_χ, with respect to the relative coordinate are introduced as

$$\beta(x\,,y) \equiv \int \frac{d^4k}{(2\pi)^3}\,r(k\,,\frac{x_0+y_0}{2})\,e^{-ik\cdot(x-y)}\,, \tag{22}$$

$$\alpha(x-y) = \int \frac{d^4k}{(2\pi)^3}\,\frac{2}{1-e^{-\beta k_0}}\,r_\chi(k)\,e^{-ik\cdot(x-y)}\,. \tag{23}$$

One obtains from the self-consistency relation

$$r(k,t) = 2\lambda^2 \int_{-\infty}^{\infty} \frac{d^4k'}{(2\pi)^4}\,\frac{r_\chi(k-k')}{1-e^{-\beta(k_0-k_0')}}\sigma(k',t)\,. \tag{24}$$

The important relation between $r(k\,,t)$ and $\sigma(k\,,t)$, eq.(24), is a result of eq.(19). The 3-momentum dependence is always even, being function of $|\vec{k}| = k$. Thus, r_χ is a function of two variables, $k_0 = \omega$ and k, hence can be written as $r_\chi(\omega\,,k)$. It is calculable from Fig.1 [4] and is odd in ω; for $\omega > 0$

$$r_\chi(\omega\,,k) = \frac{\lambda^2}{16\pi^2}\left(\sqrt{1-\frac{4m^2}{\omega^2-k^2}}\,\theta(\omega-\sqrt{k^2+4m^2}) + \frac{2}{\beta k}\,\ln\frac{1-e^{-\beta\omega_+}}{1-e^{-\beta|\omega_-|}}\right)\,, \tag{25}$$

$$\omega_\pm = \frac{\omega}{2} \pm \frac{k}{2}\sqrt{1-\frac{4m^2}{\omega^2-k^2}}\,. \tag{26}$$

The basic function g that appears in the correlator is then given by

$$g_t(\vec{k}\,,\tau) = i\int_{-\infty}^{\infty} d\omega\, H(\omega,\vec{k},t)\,e^{-i\omega\tau}\,, \tag{27}$$

$$H(k\,,t) = \frac{r_-(k\,,t)}{(k^2 - M^2(T\,,t) - \Pi(k\,,t))^2 + (\pi r_-(k\,,t))^2}\,, \tag{28}$$

$$\Pi(k\,,t) = \mathcal{P}\int_{-\infty}^{\infty} d\omega'\,\frac{r_-(\omega', \vec{k}\,,t)}{\omega' - k_0}\,, \tag{29}$$

where even and odd parts of the spectral weight are defined by

$$r_\pm(k\,,t) = r(k\,,t) \pm r(-k\,,t)\,. \tag{30}$$

The time dependence of the mass shift $M^2(T\,,t)$ is weak and its asymptotic equilibrium value $M^2(T)$ to $O[\lambda^2]$ is given later by eq.(55). With the help of

$$h_t(\omega, \vec{k}\,,\tau) = \int_0^\tau ds\, g_t(\vec{k}, s)e^{-i\omega s} = \int_{-\infty}^{\infty} d\omega'\,\frac{H(\omega', \vec{k}\,,t)}{\omega - \omega'}(e^{i\omega'\tau} - e^{i\omega\tau})\,, \tag{31}$$

the correlation function is calculated as

$$\begin{aligned}
&\langle\varphi(x)\,\varphi(y)\rangle \\
=\;& \int \frac{d^3k}{(2\pi)^3}\left[-\frac{i}{2}g_t(\vec{k}, x^0 - y^0) + \int_{-\infty}^{\infty} d\omega\,\frac{r_+(\omega, \vec{k}, t)}{2}h_t^*(\omega, \vec{k}, x^0)h_t(\omega, \vec{k}, y^0)\right. \\
&+ g_t(\vec{k}, x^0)g_t(\vec{k}, y^0)\overline{q_{\vec{k}}^2} + \dot{g}_t(\vec{k}, x^0)\dot{g}_t(\vec{k}, y^0)\overline{q_{\vec{k}^2}} \\
&\left.+\frac{1}{2}\left\{\dot{g}_t(\vec{k}, x^0)g_t(\vec{k}, y^0) + g_t(\vec{k}, x^0)\dot{g}_t(\vec{k}, y^0)\right\}\overline{\dot{q}_{\vec{k}}q_{-\vec{k}} + q_{\vec{k}}\dot{q}_{-\vec{k}}}\right] e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,.
\end{aligned} \tag{32}$$

Note that $t = (x_0 + y_0)/2$. The integrand of 3-momentum integral of this formula is divided into the antisymmetric part (1st term) and the symmetric part (the rest) under the interchange $x_0 \leftrightarrow y_0$. On the other hand, the antisymmetic part of the correlation function is related to the spectral weight H as seen from eq.(27), thus the antisymmetric part of the correlator is given by

$$\sigma_-(k\,,t) = 2\pi\, H(k\,,t)\,. \tag{33}$$

The equation thus derived (32) can be regarded as a self-consistency equation for $\langle\varphi(x)\,\varphi(y)\rangle$, or equivalently for its Fourier transform $\sigma(k\,,t)$. Its derivation rests with the Hartree self-consistency and the slow variation approximation on the central time. In the following sections we derive more convenient form of differential equations that gives an equivalent result.

4 Detailed balance and equilibrium abundance

One takes the infinite central time limit of the self-consistency equation in the preceeding section to derive detailed balance relation, from which one can analyze the equilibrium abundance.

First, one has for the correlator

$$\begin{aligned}\langle\varphi(x)\,\varphi(y)\rangle \quad &\longrightarrow \quad \langle\varphi(x)\varphi(y)\rangle_\infty = \frac{1}{2}\int\frac{d^4k}{(2\pi)^3}\,H(k,\infty)e^{-ik\cdot(x-y)} \\ &+ \quad \frac{1}{2}\int\frac{d^4k}{(2\pi)^3}\,\frac{r_+(k,\infty)}{r_-(k,\infty)}H(k,\infty)e^{-ik\cdot(x-y)}\,, \end{aligned} \tag{34}$$

in the infinite central time limit, $(x_0+y_0)/2 \to \infty$. All initial memory effects drop out in this limit, as expected. Its Fourier transform gives odd and even parts,

$$\sigma_-(k,\infty) = 2\pi H(k,\infty)\,, \quad \sigma_+(k,\infty) = \frac{2\pi r_+(k,\infty)}{r_-(k,\infty)}H(k,\infty)\,, \tag{35}$$

since the spectral function $H(k,\infty)$ is odd in k_0.

We further define a quantity,

$$\tau(k\,,\infty) = \frac{\sigma(-k,\infty)}{\sigma_-(k,\infty)}\,. \tag{36}$$

One set of the consistency equation derived by combining eqs.(35) and (36)

$$r(k,\infty)\tau(k,\infty) - r(-k,\infty)(1+\tau(k,\infty)) = 0\,, \tag{37}$$

is then written as

$$\begin{aligned} 0 \;=\; & 16\lambda^2\int_0^\infty\frac{dp_0'}{2\pi}\int\frac{d^3p'}{(2\pi)^3}\int\frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}}\int\frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}}\,(2\pi)^3\sigma_-(p',\infty) \\ & \Big\{\,\delta^{(4)}(p+p'-k_1-k_2)\,[\tau\tau'(1+f_1)(1+f_2) - (1+\tau)(1+\tau')f_1f_2] \\ & +2\delta^{(4)}(p+p'+k_1-k_2)\,[\tau\tau' f_1(1+f_2) - (1+\tau)(1+\tau')(1+f_1)f_2] \\ & +\delta^{(4)}(p+p'+k_1+k_2)\,[\tau\tau' f_1f_2 - (1+\tau)(1+\tau')(1+f_1)(1+f_2)] \\ & +\delta^{(4)}(p-p'-k_1-k_2)\,[\tau(1+\tau')(1+f_1)(1+f_2) - (1+\tau)\tau' f_1f_2] \\ & +2\delta^{(4)}(p-p'+k_1-k_2)\,[\tau(1+\tau')f_1(1+f_2) - (1+\tau)\tau'(1+f_1)f_2] \\ & +\delta^{(4)}(p-p'+k_1+k_2)\,[\tau(1+\tau')f_1f_2 - (1+\tau)\tau'(1+f_1)(1+f_2)]\,\Big\}\,. \end{aligned} \tag{38}$$

Here $\tau = \tau(p\,,\infty)\,, \tau' = \tau(p'\,,\infty)\,,$ while

$$\omega_k = \sqrt{\vec{k}^{\,2}+m^2}\,, \tag{39}$$

$$f_1 = f_{\rm th}(k_1)\,, \quad f_2 = f_{\rm th}(k_2)\,, \quad f_{\rm th}(k) = \frac{1}{e^{\omega_k/T}-1}\,. \tag{40}$$

Note that the momentum p is not integrated in this equation, thus eq.(38) is an integral equation for the unknown function $\tau(p\,,\infty)$.

There is an obvious solution for this set of equations:

$$\tau(p\,,\infty) = \frac{1}{e^{\beta p_0} - 1}\,, \tag{41}$$

due to the presence of energy conservation in each associated process of eq.(38). The 3-momentum dependence in $\tau(p\,,\infty)$ is missing. We further advance our analysis assuming that this is the unique set of solutions to the above equation. Putting these into the other set of consistency equation at infinite time, one obtains a closed form of self-consistency equation for r_-;

$$r_-(k,\infty) \;=\; 2\,\lambda^2 \int_{-\infty}^{\infty} \frac{d^4k'}{(2\pi)^3}\, H(k'\,,\infty)\, r_\chi(k+k')\frac{e^{\beta k_0}-1}{(e^{\beta(k_0+k'_0)}-1)(1-e^{-\beta k'_0})}\,, \tag{42}$$

$$H(k\,,\infty) \;=\; \frac{r_-(k,\infty)}{(k^2 - M^2(T) - \Pi(k,\infty))^2 + (\pi r_-(k,\infty))^2}\,. \tag{43}$$

The quantity $\Pi(k\,,\infty)$ is given by a principal value integral of $r_-(k\,,\infty)$ like (29). The other independent quantity r_+ is determined by

$$r_+(k\,,\infty) = \coth\frac{\beta k_0}{2}\, r_-(k\,,\infty)\,, \tag{44}$$

hence giving the correlator

$$\langle\varphi(x)\varphi(y)\rangle_\infty = \int \frac{d^4k}{(2\pi)^3}\,\frac{1}{1-e^{-\beta k_0}}\, H(k\,,\infty)\, e^{-i\,k\cdot(x-y)}\,. \tag{45}$$

This way one can completely determine $r(k\,,\infty)$, hence $\sigma(k\,,\infty)$.

One may first note the trivial case of application of eq.(45), taking the zero coupling limit $r = 0$. In this case one should understand the quantity H as

$$H(k\,,\infty) = \delta(k_0^2 - E_k^2)\,\epsilon(k_0)\,, \tag{46}$$

with

$$E_k = \sqrt{\vec{k}^2 + M^2}\,, \tag{47}$$

thus gets the free field correlator at finite temperature T.

Next, perturbative analysis of the detailed balance relation shows that to $O[\lambda^2]$

$$r_-(k\,,\infty) \;=\; 2\lambda^2 \int \frac{d^3k'}{(2\pi)^3\, E_{k'}}\left[r_\chi(k_0+E_k, \vec{k}+\vec{k}')\,\frac{e^{\beta k_0}-1}{(e^{\beta(k_0+E_{k'})}-1)(1-e^{-\beta E_{k'}})} \right.$$
$$\left. +\; r_\chi(k_0-E_k, \vec{k}-\vec{k}')\,\frac{e^{\beta k_0}-1}{(e^{\beta(k_0-E_{k'})}-1)(e^{\beta E_{k'}}-1)}\right]. \tag{48}$$

The spectral function r_- given by (48) may be written in a suggestive form by using an expression for r_χ of [9]. It is equivalent to

$$\begin{aligned}
& r_-(p\,,\infty) = \frac{\lambda^2}{2}\int d\Pi_{p'}\int d\Pi_k \int d\Pi_{k'}\,(2\pi)^3\,\delta^{(3)}(\vec{p}+\vec{p}\,'+\vec{k}+\vec{k}\,') \\
& \{\,\delta(p_0+E_{p'}-\omega_k-\omega_{k'})\,[\tilde{f}_{p'}(1+f_k)(1+f_{k'})-(1+\tilde{f}_{p'})f_kf_{k'}] \\
& +2\,\delta(p_0+E_{p'}-\omega_k+\omega_{k'})\,[\tilde{f}_{p'}(1+f_k)f_{k'}-(1+\tilde{f}_{p'})f_k(1+f_{k'})] \\
& +\delta(p_0+E_{p'}+\omega_k+\omega_{k'})\,[\tilde{f}_{p'}f_kf_{k'}-(1+\tilde{f}_{p'})(1+f_k)(1+f_{k'})] \\
& +\delta(p_0-E_{p'}-\omega_k-\omega_{k'})\,[(1+\tilde{f}_{p'})(1+f_k)(1+f_{k'})-\tilde{f}_{p'}f_kf_{k'}] \\
& +\delta(p_0-E_{p'}+\omega_k+\omega_{k'})\,[(1+\tilde{f}_{p'})f_kf_{k'}-\tilde{f}_{p'}(1+f_k)(1+f_{k'})] \\
& +2\,\delta(p_0-E_{p'}+\omega_k-\omega_{k'})\,[(1+\tilde{f}_{p'})f_k(1+f_{k'})-\tilde{f}_{p'}(1+f_k)f_{k'}]\,\}\,. \qquad (49)
\end{aligned}$$

The function $\tilde{f}_p$ was introduced here,

$$\tilde{f}_p = \frac{1}{e^{E_p/T}-1}\,, \qquad (50)$$

thermal χ particle distribution function being written as $f_k\,, f_{k'}$. The phase space factor here is

$$d\Pi_p = \frac{d^3p}{(2\pi)^3\,2E_p}\,, \qquad d\Pi_k = \frac{d^3k}{(2\pi)^3\,2\omega_k}\,. \qquad (51)$$

An important point to note is that this spectral function r_- is defined even off the mass shell $p_0 \neq \sqrt{\vec{p}^{\,2}+M^2}$. Since the negative p_0 region gives $r_-(-p_0,\vec{p}\,,\infty) = -\,r_-(p_0,\vec{p}\,,\infty)$, one should be careful in associating indivisual terms of eq.(49) with physical processes.

The correlator to $O[\lambda^2]$ is obtained using this r_- in eq.(45) along with (43). This result can be compared to a calculation using the thermal field theory in [7]. The imaginary-time formalism gives the correlator to $O[\lambda^2]$,

$$\begin{aligned}
& \langle\varphi(\tau_1,\vec{x}_1)\,\varphi(\tau_2,\vec{x}_2)\rangle_\beta \\
= \; & \Delta_\varphi(x_1-x_2) - \frac{\lambda}{2}\int_0^\beta d^4y\,\Delta_\chi(0)\Delta_\varphi(y-x_1)\Delta_\varphi(y-x_2) \\
+ \; & \frac{\lambda^2}{2}\int_0^\beta d^4y_1\;d^4y_2\,\Delta_\varphi(y_1-y_2)\Delta_\chi(y_1-y_2)\Delta_\chi(y_1-y_2)\Delta_\varphi(y_1-x_1)\Delta_\varphi(y_2-x_2) \\
+ \; & \frac{\lambda^2}{4}\int_0^\beta d^4y_1\;d^4y_2\,\Delta_\varphi(0)\Delta_\chi(y_1-y_2)\Delta_\chi(y_1-y_2)\Delta_\varphi(y_1-x_1)\Delta_\varphi(y_1-x_2) \\
+ \; & \frac{\lambda^2}{4}\int_0^\beta d^4y_1\;d^4y_2\,\Delta_\chi(0)\Delta_\chi(0)\Delta_\varphi(y_1-y_2)\Delta_\varphi(y_1-x_1)\Delta_\varphi(y_2-x_2) \qquad (52) \\
\simeq \; & \int\frac{d^3p}{(2\pi)^3}\,T\sum_n e^{-\omega_n(\tau_1-\tau_2)}e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}F(\omega_n,\vec{p})\,. \qquad (53)
\end{aligned}$$

where the variable τ is the Euclidean time defined in the range of $0 \sim \beta$, and $\omega_n = 2\pi i n/\beta \, (n = 0\,, \pm 1\,, \pm 2 \cdots)$. The quantities Δ_φ and Δ_χ are the thermal propagators of the φ and χ particles, respectively. Each term in eq.(52) corresponds to respective diagram in Fig.2. The function F here is calculated as

$$F(\omega_n, \vec{p})^{-1} = -\,\omega_n^2 + \vec{p}^{\,2} + M^2(T) + \int_{-\infty}^{\infty} d\omega\, \frac{r_-(\omega, \vec{p})}{\omega - \omega_n}\,, \tag{54}$$

$$M^2(T) = M^2 + \frac{\lambda}{24}T^2$$
$$-\,\frac{\lambda^2}{32\pi^3}(\frac{MT}{2\pi})^{1/2}e^{-M/T}\int_0^\infty dk\,\frac{k^2}{\omega_k^2}\left(\frac{e^{-\beta\omega_k}}{(1-e^{-\beta\omega_k})^2} + \frac{1}{\beta\omega_k(e^{\beta\omega_k}-1)}\right)\,, \tag{55}$$

valid at $m \ll T \ll M$ where $\omega_k = \sqrt{\vec{k}^{\,2} + m^2}$. The temperature dependent mass $M^2(T)$ has correction both of λ and λ^2 orders, but the $O[\lambda^2]$ term is Boltzmann suppressed. In the function F the proper self-energy given by a r_- integral is summed up to get a geometric form, which is allowed to $O[\lambda^2]$. The discrete energy sum here may be converted by a contour integration to the form given by the real-time correlator, which is nothing but eq.(45) along with (43). Thus, two methods give the same result.

5 Quantum kinetic equation

Derivation of the kinetic equation goes in a few steps. We shall only sketch the main point of this sequence of arguments. It starts from the self-consistency equation for the quantity $\sigma(k\,,t)$, the Fourier transformed correlator. We first note that the odd part of this quantity

$$\sigma_-(k\,,t) = \sigma(k\,,t) - \sigma(-k\,,t) \tag{56}$$

slowly varies. Thus, to order of λ^2 one can use the infinite time limit;

$$\sigma_-(k\,,t) = \sigma_-(k\,,\infty) + O[\lambda^4]\,. \tag{57}$$

The basic reason for this simplicity is due to a symmetry of certain terms of eq.(32) under $x_0 \leftrightarrow y_0$.

The even part, or a more convenient quantity

$$\tau(k\,,t) = \frac{\sigma(-k\,,t)}{\sigma_-(k\,,t)} \tag{58}$$

which is constrained by

$$\tau(k_0, \vec{k}\,, t) + \tau(-\,k_0\,\vec{k}\,, t) = -1\,, \tag{59}$$

is used for derivation of the kinetic equation. Under the condition (57) one may use the "constant" $\sigma_-(k\,, \infty)$, which can be replaced by

$$2\pi H(k\,, \infty) = \sigma_-(k\,, \infty)\,. \tag{60}$$

The kinetic equation is derived from the self-consistency relation symbolically written as

$$\tau(t) = T[\tau(t)\,, t]\,, \tag{61}$$

where the explicit and implicit (via the function τ) time dependence is written. The right hand side is understood as a functional of $\tau(t)$. By dropping higher order $O[\lambda^4]$ terms, we may derive from this equation

$$\frac{d\tau}{dt} = -\,\Gamma\ (\,\tau - T[\tau(t)\,, \infty]\,)\ , \tag{62}$$

$$\Gamma = -\,\frac{\partial}{\partial t}\,\ln\left(\,T[\tau(t)\,, t] - T[\tau(t)\,, \infty]\,\right)\ . \tag{63}$$

The neglected term is $\partial T/\partial t$ of order λ^2 and it was replaced as

$$1 - \frac{\partial T}{\partial \tau}\ \to\ 1\,, \tag{64}$$

in the left hand side. One explicitly works out this symbolic equation for our case and in the end derives the quantum kinetic equation in the form,

$$\begin{aligned}
&\frac{d\tau(p,t)}{dt} = \frac{\lambda^2}{2p_0}\int\frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}}\int\frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}}\int\frac{d^3p'}{(2\pi)^3}\int_0^\infty\frac{dp_0'}{2\pi}\,(2\pi)^4\sigma_-(p'\,,\infty) \\
&\Big\{\,\delta^{(4)}(p+p'-k_1-k_2)\,[\tau\tau'(1+f_1)(1+f_2) - (1+\tau)(1+\tau')f_1f_2] \\
&+2\delta^{(4)}(p+p'+k_1-k_2)\,[\tau\tau' f_1(1+f_2) - (1+\tau)(1+\tau')(1+f_1)f_2] \\
&+\delta^{(4)}(p+p'+k_1+k_2)\,[\tau\tau' f_1 f_2 - (1+\tau)(1+\tau')(1+f_1)(1+f_2)] \\
&+\delta^{(4)}(p-p'-k_1-k_2)\,[\tau(1+\tau')(1+f_1)(1+f_2) - (1+\tau)\tau' f_1 f_2] \\
&+2\delta^{(4)}(p-p'+k_1-k_2)\,[\tau(1+\tau')f_1(1+f_2) - (1+\tau)\tau'(1+f_1)f_2] \\
&+\delta^{(4)}(p-p'+k_1+k_2)\,[\tau(1+\tau')f_1f_2 - (1+\tau)\tau'(1+f_1)(1+f_2)]\,\Big\}\ .
\end{aligned} \tag{65}$$

In the right hand side $\tau = \tau(p\,, t)\,, \tau' = \tau(p'\,, t)$. In the final step the Markovian approximation for the rate was used; the exponential decay law for the quantity $T[\tau(t)\,, t]$ was assumed with the pole approximation.

The kinetic equation thus derived has a structural resemblance to the conventional Boltzmann equation. There are however a number of differences. The most important one is that there is no mass shell condition for the φ particle; ${p_0}^2 - \vec{p}^2 \neq M^2$. Accordingly the function τ is a function of the 4-momentum p. Related to this is that even the processes not allowed by the energy-momentum conservation on the mass shell all contribute to the collision term in the right hand side, for example $1 \leftrightarrow 3$ process such as $\varphi \leftrightarrow \varphi\chi\chi$. Even if the equilibrium solution $\tau(p\,,\infty) = 1/(e^{\beta p_0} - 1)$ discussed in the preceeding section coincides with the familiar Bose-Einstein form at $p_0 = \sqrt{\vec{p}^2 + M^2}$, the p_0 variable here is defined all the way from $-\infty$ to ∞. These are important differences, although the structural resemblance is impressive and suggests deeper understanding.

Once the kinetic equation is solved, one can compute physical quantities using the function τ. For instance, the φ energy density is calculated taking the coincident limit of two-point correlators, with due consideration of renormalization of composite operators. When Fourier transformed, the correlator is given by

$$\sigma(p\,,t) = \sigma_-(p\,,t)\,\tau(-\,p\,,t) = 2\pi\,H(p\,,t)\,\tau(-\,p\,,t)\,. \tag{66}$$

Hence

$$\begin{aligned}
\langle \mathcal{H}_\varphi \rangle &= \langle \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{M^2}{2}\varphi^2 - (\text{counter terms})\rangle \\
&= \frac{1}{2}\int \frac{d^4p}{(2\pi)^3}\,\{p_0^2 + E_p^2\}H(p\,,t)\tau(p,t) - (\text{counter terms}) \\
&= \int \frac{d^3p}{(2\pi)^3}\int_0^\infty dp_0\,\{p_0^2 + E_p^2\}H(p\,,t)\tau(p,t) \\
&+ \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int_0^\infty dp_0\,\{p_0^2 + E_p^2\}H(p\,,t) - (\text{counter terms})\,.
\end{aligned} \tag{67}$$

This is further simplified using $H(p\,,t) \approx H(p\,,\infty)$ and

$$\begin{aligned}
&\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int_0^\infty dp_0\,\{p_0^2 + E_p^2\}H(p\,,\infty) \\
= &\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int_0^\infty dp_0\,\frac{r_-(p\,,\infty)}{(p_0 + E_p)^2} + \frac{1}{2} + O[\lambda^3]\,.
\end{aligned} \tag{68}$$

Thus,

$$\langle \mathcal{H}_\varphi \rangle \simeq \int \frac{d^3p}{(2\pi)^3}\int_0^\infty dp_0\,\{p_0^2 + E_p^2\}H(p\,,\infty)\tau(p,t) + c\,\lambda^2\frac{T^6}{M^2}$$

$$\simeq \int \frac{d^3p}{(2\pi)^3}\, E_p\, \tau(E_p, \vec{p}\,, t) + c\,\lambda^2 \frac{T^6}{M^2}\,, \tag{69}$$

$$c = \frac{1}{69120} \sim 1.4\, \times 10^{-5}\,, \tag{70}$$

dropping Boltzmann suppressed $O[\lambda^2]$ terms. In the last step we replaced the spectral weight H by the delta function, since perturbative analysis works for this quantity. In this computation we assumed the mass hierarchy for the light χ particle, $m \ll T$. We used in this derivation the formula (49) to get ($T \ll M$ assumed)

$$\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int_0^\infty dp_0\, \frac{r_-(p\,,\infty)}{(p_0+E_p)^2} + \frac{1}{2} - (\text{counter terms})$$
$$\simeq 2\lambda^2 \int d\Pi_k \int d\Pi_{k'}\, (k^2 + k'^{\,2})\, f_k f_{k'} \int d\Pi_p \int d\Pi_{p'}\, \frac{(2\pi)^3 \delta^{(3)}(\vec{p} + \vec{p}\,' + \vec{k} + \vec{k}\,')}{(E_p + E_{p'})^3}$$
$$\sim \frac{\lambda^2}{32\pi^2}\int d\Pi_k \int d\Pi_{k'}\, (k^2 + k'^{\,2})\, f_k f_{k'} \int_0^\infty dp\, \frac{p^2}{E_p^5} = c\,\lambda^2 \frac{T^6}{M^2}\,, \tag{71}$$

dropping Botzmann suppressed minor terms. Physical processes that contribute to this equilibrium result are

$$\chi\chi \rightarrow \varphi\varphi'\,, \quad \chi \rightarrow \varphi\varphi'\chi\,, \tag{72}$$

φ' being the partner φ particle. All other channels are suppressed by the Boltzmann factor,

The time dependence of physical quantities appears via the function $\tau(E_p, \vec{p}\,, t)$ in eq.(69). This function defined on the mass shell ($E_p = \sqrt{\vec{p}^{\,2} + M^2}$) follows the usual Boltzmann equation, as seen from the kinetic equation (65) along with the approximation,

$$\sigma_-(p\,,\infty) \approx \frac{2\pi}{2E_p}\,\delta(p_0 - E_p)\,. \tag{73}$$

It is thus clear that in a state near thermal equilibrium $\tau(E_p, \vec{p}\,, t) \sim 1/(e^{E_p/T} - 1)$. Hence for $T/M \ll 1/\sqrt{c}\lambda \approx 3 \times 10^2/\lambda$

$$\langle \mathcal{H}_\varphi \rangle \sim \langle \mathcal{H}_\varphi \rangle_{\rm eq} = \int \frac{d^3p}{(2\pi)^3}\, \frac{\sqrt{p^2 + M^2}}{e^{\sqrt{p^2+M^2}/T} - 1} + c\,\lambda^2 \frac{T^6}{M^2}\,. \tag{74}$$

The same equilibrium result was derived using the thermal field theory in [7]. Derivation given here is a separate and independent confirmation of the equilibrium density from the kinetic approach. The temperature power thus derived ($\propto T^6$) differs from that given in ref.[3], which should be corrected.

As discussed in [7], the time evolution equation in cosmology is approximately given by (at $m \ll T \ll M$)

$$\frac{dn_{\varphi}}{dt} + 3H\, n_{\varphi} = -\, \langle \sigma v \rangle \, \left(n_{\varphi}^2 - n_{\rm eq}^2 \right) \,, \tag{75}$$

$$n_{\rm eq} \sim (\frac{MT}{2\pi})^{3/2} e^{-M/T} + c\, \lambda^2 \frac{T^6}{M^3} \,, \tag{76}$$

where H is the Hubble rate $\sim 1.7\sqrt{N}\, T^2/m_{\rm pl}$. Numerical analysis shows that the picture of the sudden freeze-out [2] holds such that until the freeze-out the number density follows the equilibrium value $n_{\rm eq}$. Thus, the most important part of cosmological application is to estimate the freeze-out temperature using the new equilibrium number density. This estimate is done by equating the annihilation rate $\langle \sigma v \rangle \, n_{\rm eq}$ to the Hubble rate H.

The close connection with the equilibrium thermal field theory is made evident by taking the infinite central time limit; the non-equilibirum correlator defined here approaches the equilibrium value given by the thermal field theory,

$$\langle \varphi(x) \varphi(y) \rangle = {\rm tr}\; \rho_i \, \varphi(x) \varphi(y) \;\; \rightarrow \;\; {\rm tr}\; e^{-\beta\, H_{\rm tot}}\, \varphi(x)\varphi(y) / {\rm tr}\; e^{-\beta\, H_{\rm tot}} \,. \tag{77}$$

The equilibrium correlator is time-translation invariant, hence the central time dependence disappears for the correlator and the non-equilibrium correlator recovers this invariance. Moreover, our formula for the correlator and physical quantities such as the energy density indicates that the approach towards thermal equilibrium described by the first term in eq.(69) is essentially determined by a thermally averaged rate of on-shell quantities, consistent with the previous estimate of this quantity in derivation of the kinetic equation.

Thus, the most important change for cosmological estimate of the relic mass density is the new equilibrium quantity. This equilibrium value is precisely given by the Gibbs formula, using the total hamiltonian, not its free part. This is the main reason the use of thermal field theory in our companion paper [7] is justified in computation of the relic mass density. It has to be kept in mind that the simple ideal gas form of distribution function must be questioned in the low temperature region. Even a higher order term in coupling is more important than the zero-th order Boltzmann suppressed term when the Boltzmann suppression factor is huge. The appearance of our temperature power term for the equilibrium number density is due to a continuous integral of these Boltzmann suppressed terms which dominates over a single contribution of quasi-particle object given by approximating a Breit-Wigner form of integral by a single complex pole.

Despite this rather obvious change, the thermally averaged Boltzmann equation is hard to justify at low and intermediate temperatures, or its slight modification is difficult to lead to the relevant correction. Although our work justifies in some sense the Boltzmann equation at high temperatures, it should be kept in mind that full quantum mechanical treatment is essential to derive the correct low temperature result.

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Figure caption

Fig.1

Diagram for the spectral weight of two light χ particle intermediate states denoted by broken lines.

Fig.2

Diagram for thermal correlator to $O[\lambda^2]$. The solid and broken lines correspond to the φ and χ particle propagators in thermal equilibrium, respectively.